UNIVERSITÄT BONN
HRL
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HUMANOID ROBOTS LAB
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Humanoid Robotics
Submission until 18.06.2026 (10:00).
Discussion in Tutorial on 23.06.2026.
Important: Solution should be **type-written** and handed in as pdf, handwritten solutions are **not** accepted. The solution should be uploaded under the name Assignment06_solution.pdf in the main branch of the assigned git repository.

Assignment 6 (Graded) (Total Points: 30)

1. Inverse Kinematics of a 2-Link Planar Arm (9 points)

Question: Consider a planar robot arm with two revolute joints and link lengths $l_1 > 0$ and $l_2 > 0$. The joint angles are q_1 and q_2. The end-effector position is $p = [x, y]^\top$. The forward kinematics are given by

$$x = l_1 \cos q_1 + l_2 \cos(q_1 + q_2),$$
$$y = l_1 \sin q_1 + l_2 \sin(q_1 + q_2).$$

1. Starting from the given forward kinematics, derive an expression for $\cos(q_2)$ in terms of x, y, l_1, and l_2.

   (1 point)

2. State the condition on $cos(q_2)$ for real-valued IK solutions. Then derive the 2 possible analytical solution for q_2 and explain their geometric meaning.

   (2 points)

3. Derive the corresponding expression for q_1 for each valid value of q_2.

   (2 points)

4. State the condition under which a Cartesian point (x, y) is reachable. Express the condition using $r = \sqrt{x^2 + y^2}$, l_1, and l_2.

   (1 point)

5. Derive the manipulator Jacobian

$$J(q) = \frac{\partial(x, y)}{\partial(q_1, q_2)}.$$

   (2 points)

6. State one configuration where the Jacobian is singular.

   (1 point)

2. Motion Planning in Configuration Space (5 points)

Question: Consider a simplified two-dimensional configuration space

$$C = [0, 10] \times [0, 10].$$

A robot is represented as a point in this configuration space. The start configuration is

$$q_I = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and the goal configuration is

$$q_G = \begin{bmatrix} 9 \\ 8 \end{bmatrix}.$$

The obstacle region in configuration space is the closed axis-aligned rectangle

$$C_{\text{obs}} = [4, 6] \times [3, 7].$$

Since the obstacle is closed, configurations on the boundary of this rectangle are also considered to be in collision. All configurations outside this obstacle region are in

$$C_{\text{free}} = C \setminus C_{\text{obs}}.$$

An RRT is initialized with the following vertices:

$$q_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad q_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \qquad q_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

The planner samples

$$q_{\text{rand}} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}.$$

Use the Euclidean distance

$$\rho(q_a, q_b) = \|q_a - q_b\|_2.$$

1. Determine whether q_1, q_2, and q_{rand} lie in C_{free} or C_{obs}.

   (1 point)

2. Compute the Euclidean distance from q_{rand} to each existing RRT vertex q_0, q_1, and q_2. Which vertex is selected as q_{near}?

   (1 point)

3. The local planner tries to connect q_{near} to q_{rand} using the straight-line segment

   $$q(s) = q_{\text{near}} + s(q_{\text{rand}} - q_{\text{near}}), \qquad s \in [0, 1].$$

   Check whether this segment intersects C_{obs}. If it intersects, compute s_{entry} the first value of s at which the segment enters the obstacle.

   (2 points)

4. Instead of adding q_{rand}, the planner adds the last safe configuration before collision using

   $$q_s = q_{\text{near}} + s_s(q_{\text{rand}} - q_{\text{near}}),$$

   where

   $$s_s = s_{\text{entry}} - 0.1.$$

   Compute q_s.

   (1 point)

3. Trajectory Generation (4 points)

Question: A robot has two joints J1 and J2. J1's acceleration is twice that of J2, whereas J2's maximum speed is three times that of J1. J1 and J2 are to cover the same joint angle distance.

1. For un-synchronized constant velocity motion, what is the ratio of time taken by J1 to J2?

   (1 point)

2. For un-synchronized symmetric trapezoidal velocity motion, assume that each joint accelerates with constant positive acceleration, moves at constant velocity, and then decelerates with the same acceleration magnitude. If J2's acceleration and speed are

   $$10^\circ/\mathrm{s}^2 \quad \text{and} \quad 30^\circ/\mathrm{s},$$

   respectively, how much time will J1 and J2 take to cover a distance of 2π radians?

   (2 points)

3. If multi-joint synchronized trapezoidal motion were to be performed, which joint determines the synchronization time, and which joint will not reach its maximum speed? Explain briefly.

   (1 point)

4. Trajectory Optimization: Numerical Solver Behaviour (12 points)

Question: This question studies the numerical behaviour of local optimization methods used in trajectory optimization. The variables below can be interpreted as low-dimensional trajectory parameters or local updates of a larger trajectory vector.

1. Consider the ill-conditioned quadratic objective

   $$F(x) = 50x_1^2 + \frac{1}{2}x_2^2, \qquad x^{(0)} = \begin{bmatrix}1\\1\end{bmatrix}.$$

   Perform two iterations of gradient descent using

   $$x^{(k+1)} = x^{(k)} - \alpha \nabla F(x^{(k)}), \qquad \alpha = 0.05.$$

   Compute $x^{(1)}$ and $x^{(2)}$. Then state the stability condition for gradient descent along the x_1 direction and check whether the chosen step size satisfies it. What numerical behaviour do you observe in the x_1 direction?

   (3 points)

2. Now consider the one-dimensional non-convex objective

   $$F(s) = s^4 - 3s^2, \qquad s^{(k)} = 0.5.$$

   Compute the gradient $g = F'(s^{(k)})$ and Hessian $H = F''(s^{(k)})$. Then compute the undamped Newton step

   $$\Delta s_N = -H^{-1}g.$$

   Compute the new point $s^{(k+1)} = s^{(k)} + \Delta s_N$ and compare $F(s^{(k+1)})$ with $F(s^{(k)})$. What went wrong?

   (3 points)

3. A trust-region method instead solves the local quadratic subproblem

$$\min_{\Delta s} \quad m(\Delta s) = g\Delta s + \frac{1}{2}H\Delta s^2$$

subject to

$$|\Delta s| \leq 0.5.$$

Using the values of g and H from part 2, solve this one-dimensional trust-region subproblem. Compute the chosen update and the new cost.

(2 points)

4. A Levenberg–Marquardt-style damped update uses

$$\Delta s_{\text{LM}} = -(H+\lambda)^{-1}g, \qquad \lambda = 10.$$

Compute Δs_{LM}, the new point, and the new cost. Explain briefly why damping helps in this case and compare the LM update with the trust-region update from part 3.

(2 points)

5. Finally, consider a local QP arising from a linearized collision constraint. The optimizer would like to take the unconstrained update

$$\Delta q_{\text{goal}} = -0.4,$$

represented by the quadratic objective

$$\min_{\Delta q} \quad \frac{1}{2}(\Delta q + 0.4)^2.$$

At the current trajectory waypoint, the signed distance to an obstacle is

$$d(q^{(k)}) = 0.10,$$

the required safety distance is

$$d_{\text{safe}} = 0.30,$$

and the local signed-distance gradient is

$$\nabla_q d(q^{(k)}) = 1.$$

Therefore, the linearized distance model is

$$d(q^{(k)} + \Delta q) \approx d(q^{(k)}) + \nabla_q d(q^{(k)})\Delta q = 0.10 + \Delta q.$$

The trust region is

$$|\Delta q| \leq 0.5.$$

Form the linearized collision constraint and solve the resulting one-dimensional QP.

(2 points)