Fundamentals of Manipulation

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Goal of This Chapter

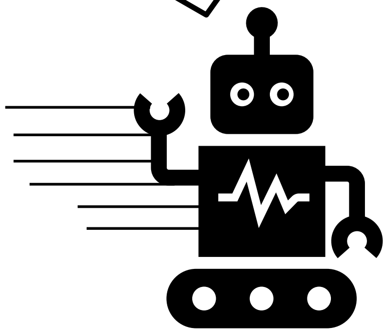
- Learn the fundamentals of robotic manipulation
- Understand the concept of **kinematic chains**
- Learn how to calculate **forward** and **inverse kinematics**
- Understand **Denavit-Hartenberg parameters**
- Understand how to use **reachability maps** to compute the robot's **kinematic capabilities**

Why Is Manipulation Needed?

- To act appropriately, robots must **observe** the world as shown in prior lectures
- However, **passively** observing the world is not enough
- Smart robots must integrate **perception and action** to gain relevant information and execute tasks

Hey robot, get me the tomato soup

OK, let's search for it

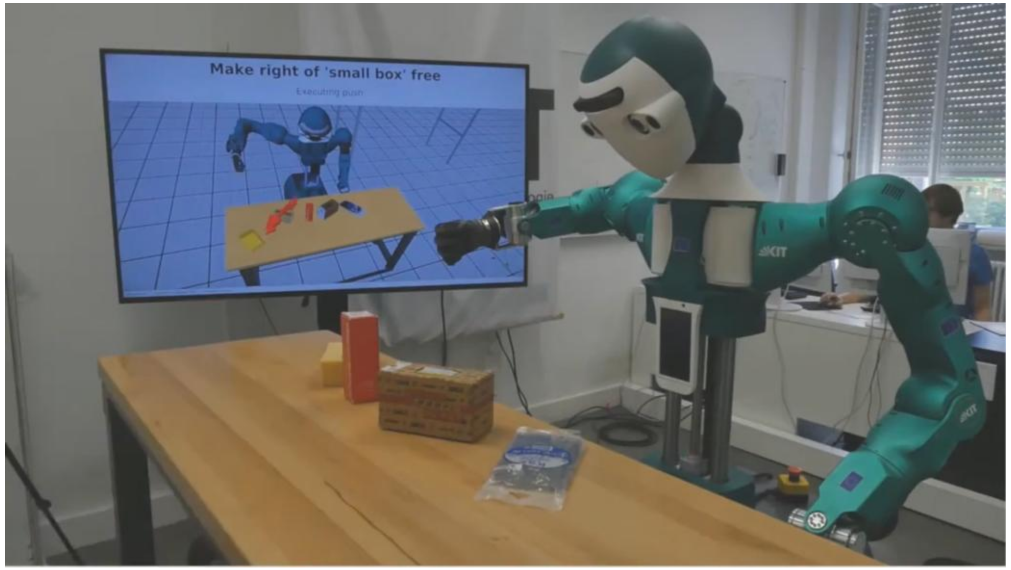


Use Manipulation Actions to...

... look behind occluding objects

Yao et al., ICRA 2025

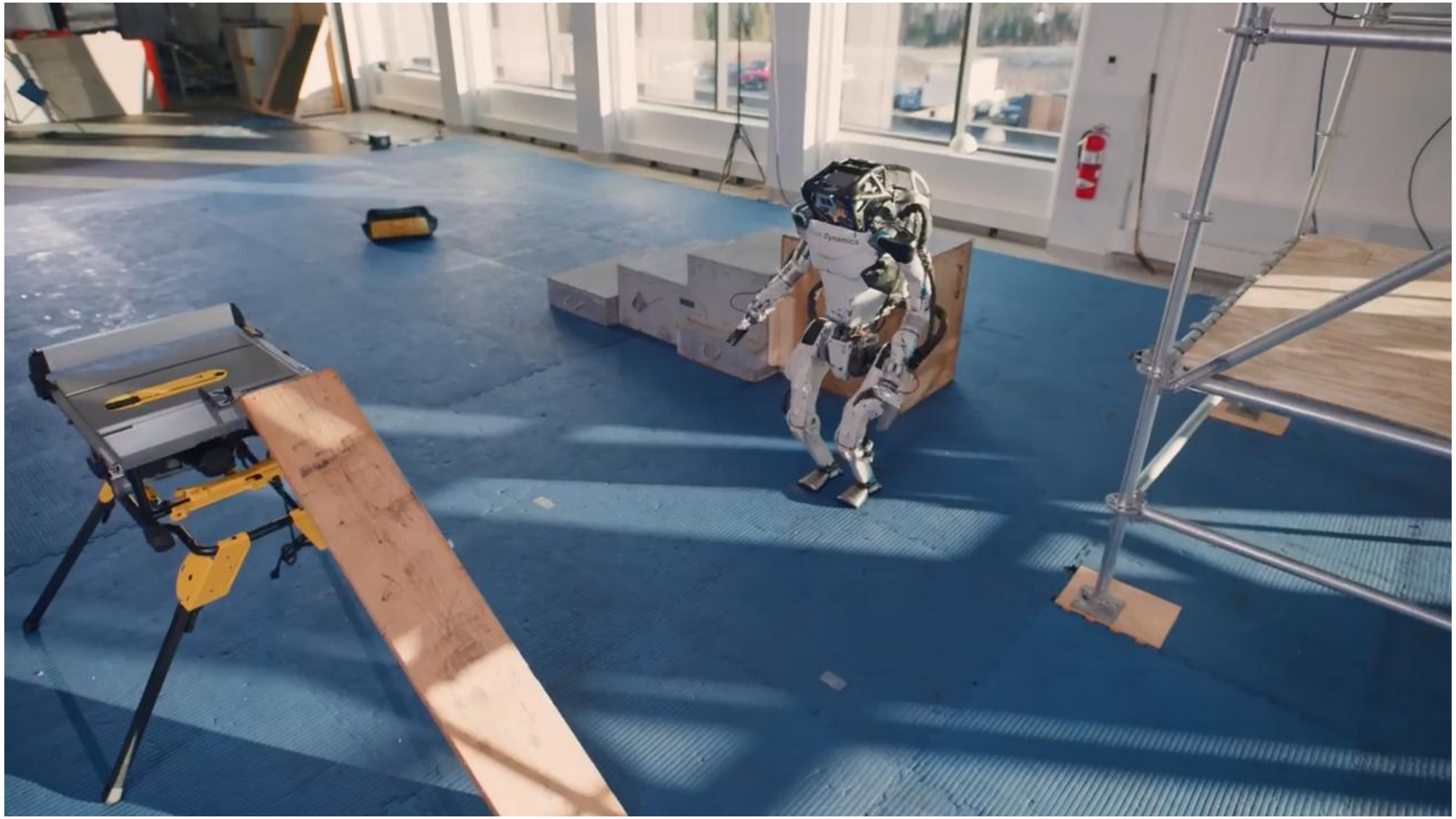
... re-arrange objects



Paus et al., ICRA 2020

Use Manipulation Actions to…

… combine with whole-body body motion planning

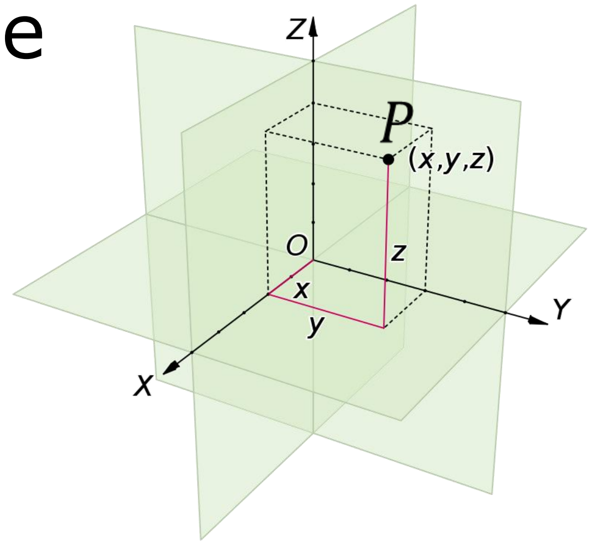


Boston Dynamics: https://www.youtube.com/watch?v=-e1_QhJ1EhQ

Recap: Transformation Basics

- **Cartesian Coordinates:**
  One way to represent the position in 3D space

$$P = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$



- **Homogeneous Coordinates**
  3-dimensional point is represented by a 4-dimensional vector

$$P = \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

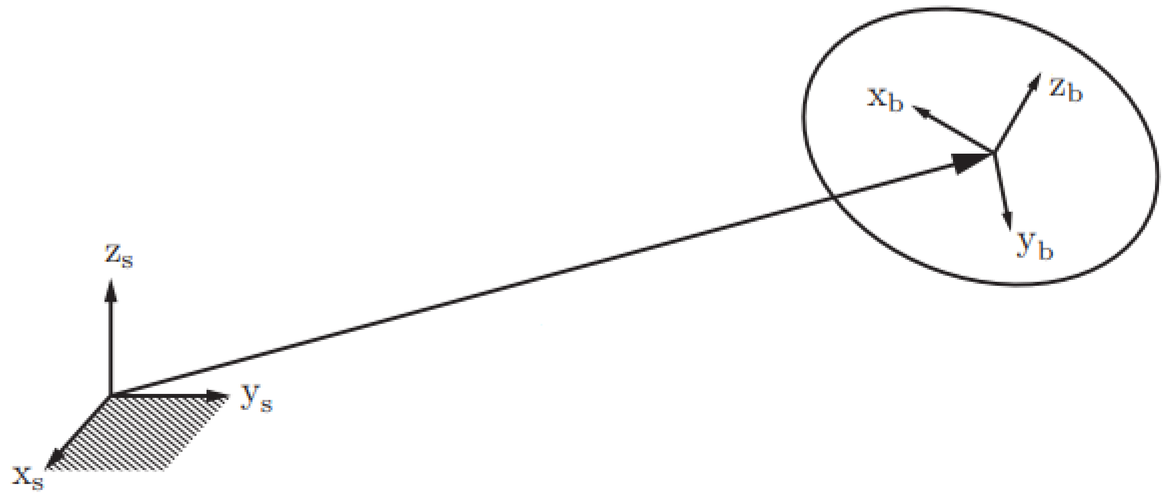
Recap: Transformation Basics

- Rigid body pose in 3D consists of **translation** and a **rotation**
- Using homogeneous coordinates, both can be represented in a single transform matrix.

Translation

- Translation of a rigid body from A to B can be described by a 3D vector

$$r_{AB} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$



Recap: Transformation Basics

- Rigid body pose in 3D consists of **translation** and a **rotation**
- Using homogeneous coordinates, both can be represented in a single transform matrix.

Rotation

- 3x3 matrix to represent the rotation of a rigid body in 3-dimensional space
- Two important properties:
  - $R^{-1} = R^T \rightarrow R * R^T = I$
  - $\det(R) = 1$

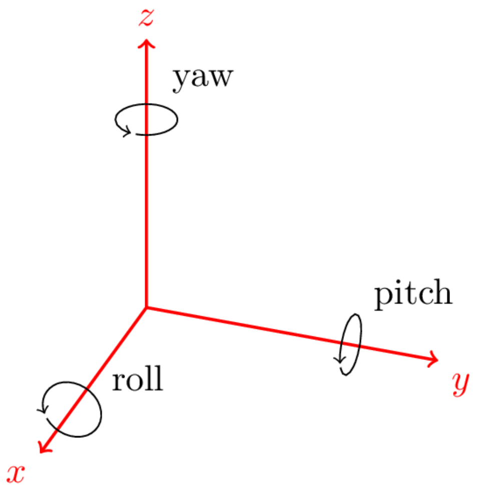
Recap: Transformation Basics

Euler Angles

- **One way** to represent a general rotation of a rigid body
- The rotation is described by a chain of rotations around 3 different axes
- We can obtain a general rotation matrix by using **matrix multiplication**

Recap: Transformation Basics

Example:
Using roll-pitch-yaw angles and **matrix multiplication**



$$R = R_z(\theta_z)\, R_y(\theta_y)\, R_x(\theta_x)$$

$$R = \begin{bmatrix} \cos(\theta_z) & -\sin(\theta_z) & 0 \\ \sin(\theta_z) & \cos(\theta_z) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta_y) & 0 & \sin(\theta_y) \\ 0 & 1 & 0 \\ -\sin(\theta_y) & 0 & \cos(\theta_y) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_x) & -\sin(\theta_x) \\ 0 & \sin(\theta_x) & \cos(\theta_x) \end{bmatrix}$$

Recap: Transformation Basics

Homogeneous Transformation Matrix

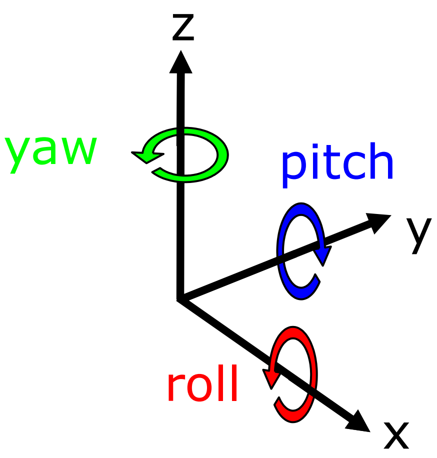
- Encodes the transformation (i.e., rotation and translation) of a rigid body in 4x4 matrix
- Element of SE(3)

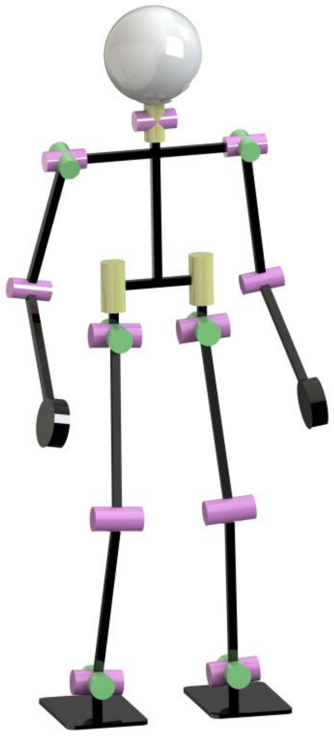
$$T = \begin{bmatrix} [R]_{3X3} & \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} \quad \rightarrow \quad T = \begin{bmatrix} R & P \\ 0 & 1 \end{bmatrix}$$

- The inverse can be calculated as: $T^{-1} = \begin{bmatrix} R^T & -R^T P \\ 0 & 1 \end{bmatrix}$

Kinematics

- The humanoid body is relatively complex, it has more than 20 degrees of freedom





Neck: yaw, pitch

Shoulder: roll, pitch

Elbow: pitch

Hip: roll, pitch, yaw

Knee: pitch

Ankle: roll, pitch

NimbRo-OP, AIS, Uni Bonn

Kinematic Chains

- Complex bodies are organized in **kinematic chains** of rigid links that are connected by joints

Leg chain

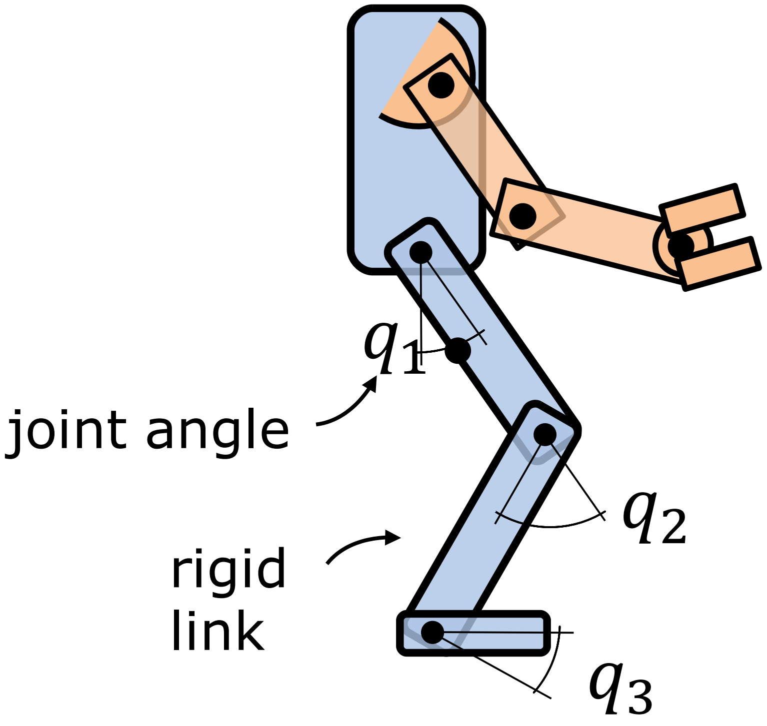
hip

↓

knee

↓

ankle



Arm chain

shoulder

↓

elbow

↓

wrist

Types of Kinematic Chain

- Open kinematic chain (e.g., robotic arm)
- Closed kinematic chain (e.g., delta robot)
- Semi-closed kinematic chain (e.g., bimanual manipulation)

Courtesy: Universal Robots

Courtesy: Fanuc

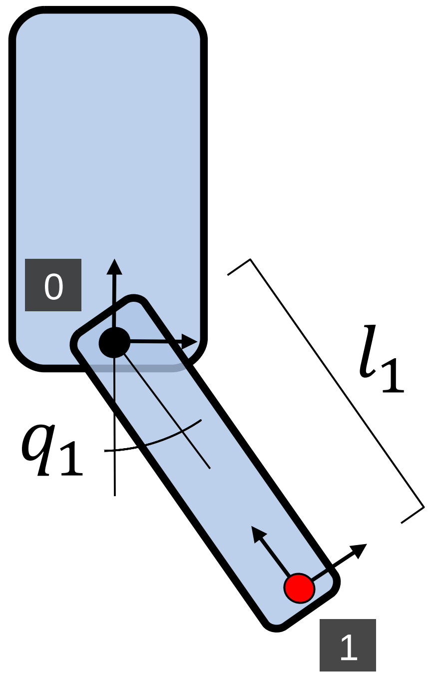
Courtesy: Clover Lab

Kinematic Parameters

- Rigid body transformations can be described by **translation followed by a rotation**
- Each link of the kinematic chain is transformed relative to its parent link
- Each joint can be explained by:
  - **Joint parameter** (i.e., rotation)
  - **Relative transformation** to other joints

$$P^0 = \begin{bmatrix} P_x^0 \\ p_y^0 \end{bmatrix}$$

$$P^1 = ?$$

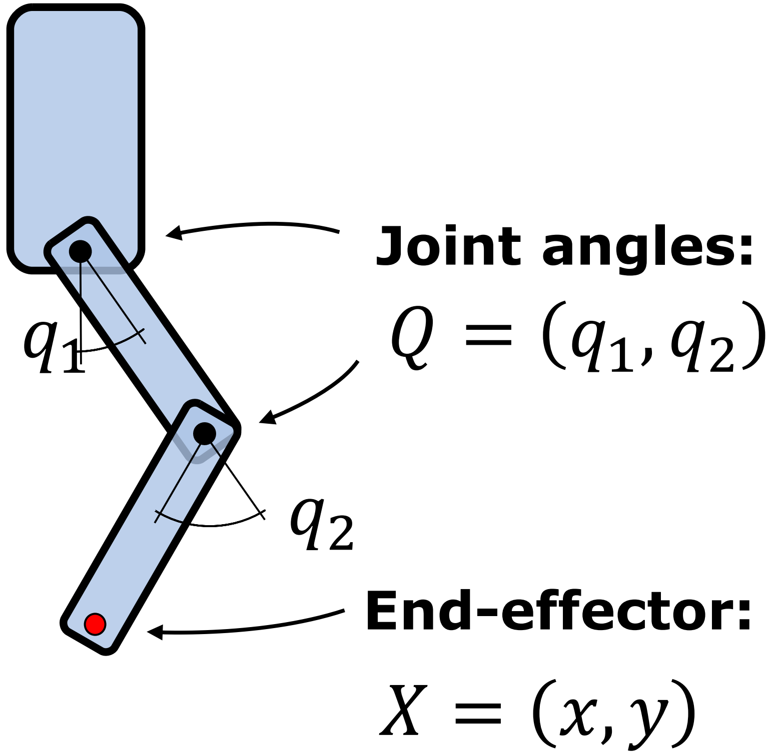


$$P^1 = \begin{bmatrix} cosq_1 & -sinq_1 \\ sinq_1 & cosq_1 \end{bmatrix} \left(P^0 + \begin{bmatrix} 0 \\ -l_1 \end{bmatrix} \right)$$

Kinematics

- Given an end-effector (EE) pose what are possible joint angles to reach it?
- Given the joint angles, what is the end-effector pose?

- **Forward Kinematics:** $X = f(Q)$
- **Inverse Kinematics:** $Q = f^{-1}(X)$



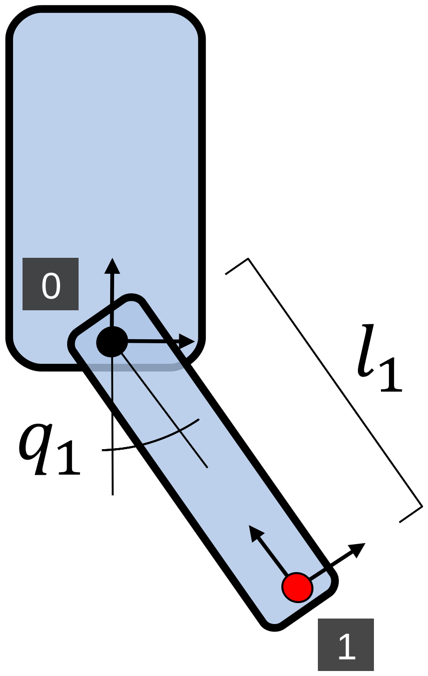
Forward Kinematics

- **Homogeneous coordinates** to represent the translation and the rotation as a matrix multiplication

$$P^0 \Rightarrow \begin{bmatrix} P_x^0 \\ P_y^0 \\ 1 \end{bmatrix} \quad P^1 \Rightarrow \begin{bmatrix} P_x^1 \\ P_y^1 \\ 1 \end{bmatrix}$$

$$P^1 = \begin{bmatrix} cosq_1 & -sinq_1 \\ sinq_1 & cosq_1 \end{bmatrix} \left(P^0 + \begin{bmatrix} 0 \\ -l_1 \end{bmatrix} \right)$$

$$\boxed{\begin{bmatrix} P_x^1 \\ P_y^1 \\ 1 \end{bmatrix} = \begin{bmatrix} cosq_1 & -sinq_1 & 0 \\ sinq_1 & cosq_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -l_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} P_x^0 \\ P_y^0 \\ 1 \end{bmatrix}}$$



Forward Kinematics

- Combine the translation and the rotation into one transformation matrix and use a **symbolic notation**

$$P^1 = R(q_1) \cdot \mathrm{t}(l_1) \cdot P^0$$

$$\begin{bmatrix} P_x^1 \\ P_y^1 \\ 1 \end{bmatrix} = \begin{bmatrix} cosq_1 & -sinq_1 & 0 \\ sinq_1 & cosq_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -l_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} P_x^0 \\ P_y^0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} P_x^1 \\ P_y^1 \\ 1 \end{bmatrix} = \begin{bmatrix} cosq_1 & -sinq_1 & l_1 sinq_1 \\ sinq_1 & cosq_1 & -l_1 cosq_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} P_x^0 \\ P_y^0 \\ 1 \end{bmatrix}$$

To

$$\boldsymbol{P^1 = T_0^1(q_1, l_1) \cdot P^0}$$

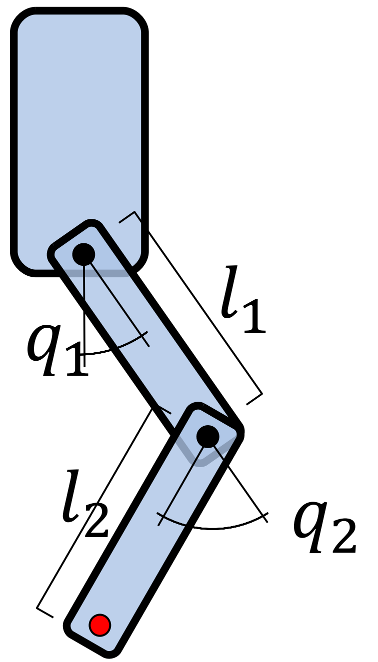
From

Forward Kinematics

- Now, transformations can be easily **concatenated**
- The order of the transformations follows the hierarchy of the kinematic chain from root to leaf

$$P^2 = T_1^2(q_2, l_2) \cdot T_0^1(q_1, l_1) \cdot P^0$$

$$P^2 = T_0^2 \cdot P^0$$



$$P^0 = \begin{bmatrix} P_x^0 \\ P_y^0 \end{bmatrix}$$

$P^2 = ?$

Inverse Kinematics (IK) and why do we need it?

- Robot actuators are controlled in **joint space**
- Manipulation tasks are usually specified in **end-effector space**
- IK computes the **joint angle values** so that the **end-effector reaches a desired pose**
- IK is challenging and cannot be as easily computed as FK
- There might exist several possible solutions, or there may be no solution at all

Inverse Kinematics (IK)

- Many different approaches to solving IK problems exist
- **Analytical methods**: closed-form solution
- **Numerical methods**: iteratively calculate a sequence of configurations that approach target pose

Analytical IK Methods

- Closed-form solution (using trigonometry, geometry)
- Computes all IK solutions, determines whether or not a solution exists
- Once the equations are derived, solutions are **very fast to compute**
- No need to define solution parameters or initial guesses
- Often **difficult to define**, must be derived
- Individual equations for robots with different kinematic structures

Numerical IK Methods

- Given a start configuration, **iteratively calculate a joint configuration** corresponding to the end-effector pose
- Includes different classes of methods, e.g.:
  - **Differential IK** using the Jacobian
  - **Optimization**-based **IK** by minimizing pose error
- Usually **much slower but also more general**
- Usually **slower than analytical IK**, but also **more general and flexible**

Jacobian of a Vector Function

- Let $f: \mathbb{R}^n \to \mathbb{R}^m, y = f(x)$
- Jacobian of f at x is the matrix of first-order partial derivatives is given by $J_f(x) = \begin{bmatrix} \frac{\delta f_1}{\delta x_1} & \cdots & \frac{\delta f_1}{\delta x_n} \\ \vdots & \ddots & \vdots \\ \frac{\delta f_m}{\delta x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$
- It gives a local first-order approximation:
$$f(x + \Delta x) \approx f(x) + J_f(x)\Delta x$$
- It tells us how sensitive each output is to each input.

Differential Inverse Kinematics

- **Differential IK** is a class of numerical IK methods based on local linearization of forward kinematics
- Small changes in joint space produce approximate small changes in task space

$$\Delta x \approx J(q)\Delta q$$

- At each iteration, we compute a joint update Δq to reduce the current end-effector error
- In practice, differential IK is **Jacobian**-based

Jacobian-based Differential IK Methods

- Jacobian **inverse (J^{-1}):** square, non-singular Jacobian required

$$\Delta q = J^{-1}\Delta x$$

- Jacobian **pseudoinverse (J^{+}):** least-squares solution, also works for non-square Jacobians

$$\Delta q = J^{+}\Delta x$$

- Jacobian **transpose (J^{T}):** simple gradient-like update, with step size $\alpha > 0$

$$\Delta q = \alpha J^{T}\Delta x$$

- **Damped least squares:** robust near singularities

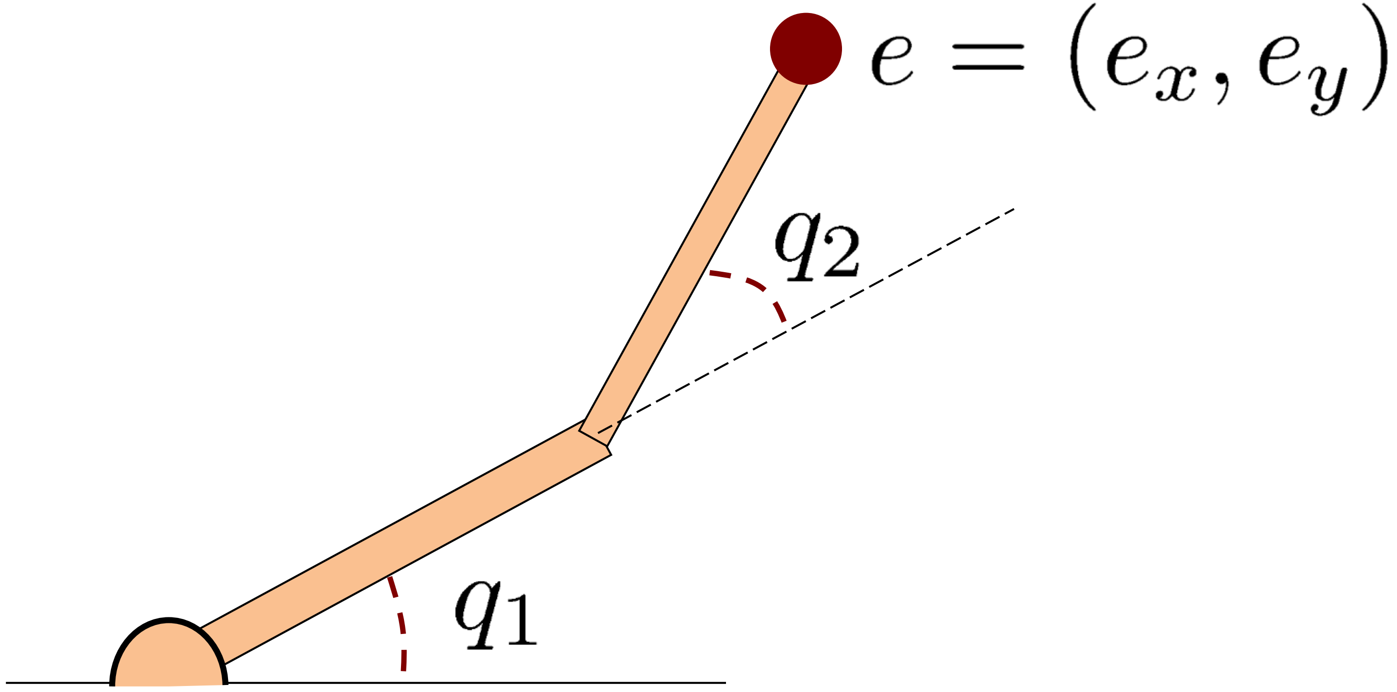
$$\Delta q = J^{T}(JJ^{T} + \lambda^2 I)^{-1}\Delta x$$

Singularities in Differential IK

- A singularity is a configuration where the Jacobian loses rank
- In such configurations, some end-effector motions are not achievable
- Near singularities, very small desired task-space changes can require very large joint changes
- This makes Jacobian inverse methods unstable or ill-conditioned
- Damped least squares improves robustness near singularities

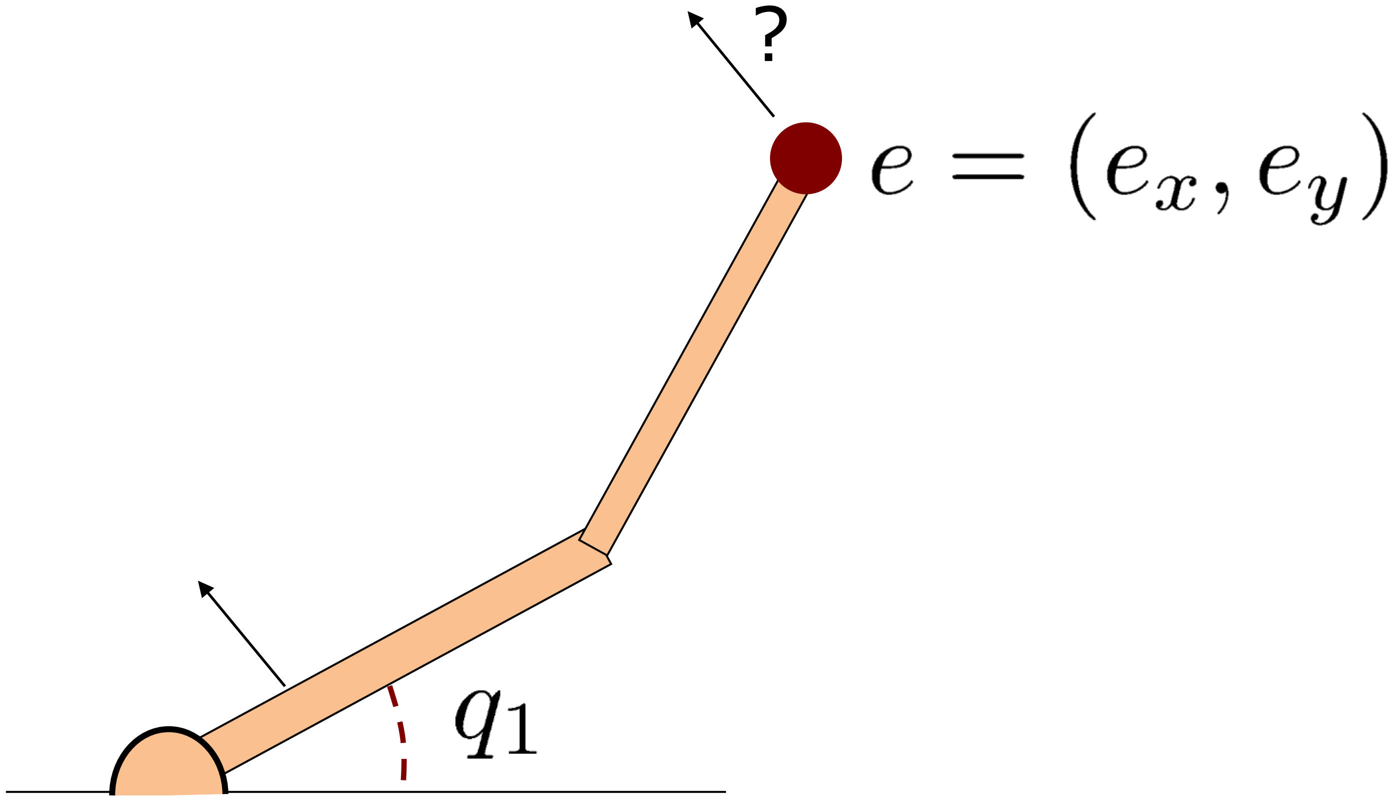
Inverse Kinematics: Example

- Consider a simple 2D robot arm with two 1-DOF joints
- Given a desired end-effector pose $\mathbf{e}$
- Compute joint angles q_1 and q_2



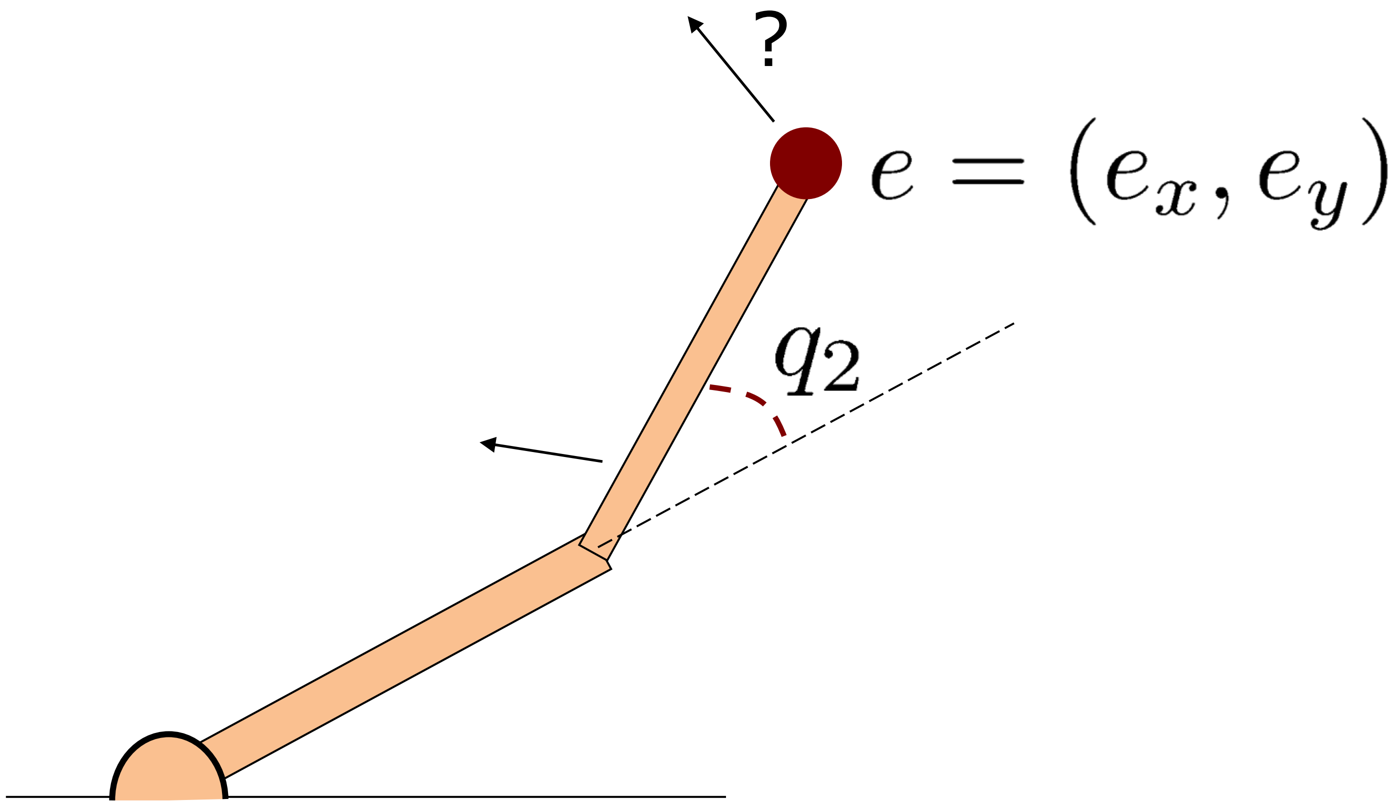
Inverse Kinematics: Example

- If we increased q_1 by a small amount, what would happen to $\mathbf{e}$?



Inverse Kinematics: Example

- If we increased q_2 by a small amount, what would happen to $\mathbf{e}$?



Numerical Approach Using the Jacobian

- Jacobian matrix for this simple example would look like:

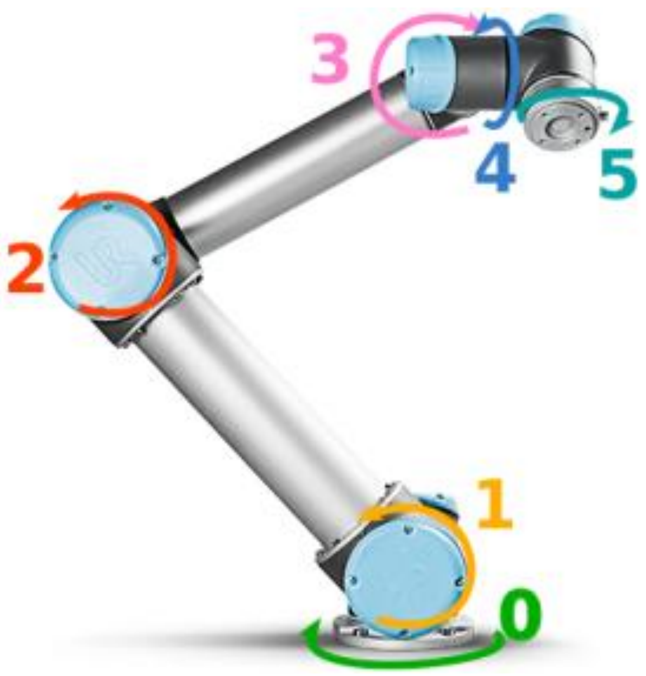
$$\mathbf{J}(\mathbf{e},\mathbf{q}) = \begin{pmatrix} \frac{\partial e_x}{\partial q_1} & \frac{\partial e_x}{\partial q_2} \\ \frac{\partial e_y}{\partial q_1} & \frac{\partial e_y}{\partial q_2} \end{pmatrix}$$

- Defines how each component of $\mathbf{e}$ changes w.r.t. joint angle changes
- For any given vector of joint values, we can compute the components of the Jacobian

Numerical Approach Using the Jacobian

- Usually, the Jacobian will be an 6x*N* matrix where *N* is the number of joints
- Jacobian can be computed based on the equations of FK

$$\mathbf{J}(\mathbf{e},\mathbf{q}) = \begin{pmatrix} \frac{\partial e_x}{\partial q_0} & \cdots & \frac{\partial e_x}{\partial q_5} \\ \frac{\partial e_y}{\partial q_0} & \cdots & \frac{\partial e_y}{\partial q_5} \\ \frac{\partial e_z}{\partial q_0} & \cdots & \frac{\partial e_z}{\partial q_5} \\ \frac{\partial e_\phi}{\partial q_0} & \cdots & \frac{\partial e_\phi}{\partial q_5} \\ \frac{\partial e_\theta}{\partial q_0} & \cdots & \frac{\partial e_\theta}{\partial q_5} \\ \frac{\partial e_\psi}{\partial q_0} & \cdots & \frac{\partial e_\psi}{\partial q_5} \end{pmatrix}$$



Numerical Approach Using the Jacobian

- Given a desired incremental change in the end-effector configuration, we can compute the corresponding incremental change of $\mathbf{q}$:

$$\mathbf{J}\Delta\mathbf{q} = \Delta\mathbf{e}$$
$$\Delta\mathbf{q} = \mathbf{J}^{-1}\Delta\mathbf{e}$$

- As $\mathbf{J}$ cannot be inverted in the general case, it is replaced by the pseudoinverse or by the transpose in practice

Numerical Approach Using the Jacobian

- Forward kinematics is a **nonlinear function**
- Thus, we have an approximation that is only valid near the current configuration
- Until the end-effector is close to the desired pose, repeat:
  - Compute the Jacobian
  - Take a small step towards the goal

End-Effector Goal and Step Size

- Let $\mathbf{e}$ represent the current end-effector pose and $\mathbf{g}$ represent its desired goal pose
- Choose a value for $\Delta\mathbf{e}$ that will move $\mathbf{e}$ closer to $\mathbf{g}$, theoretically:

$$\Delta\mathbf{e} = \mathbf{g} - \mathbf{e}$$

- But non-linearity prevents the end-effector to reach the goal exactly
- Thus, to avoid oscillation, take a smaller step:

$$\Delta\mathbf{e} = \alpha(\mathbf{g} - \mathbf{e}), 0 \leq \alpha \leq 1$$

Basic Jacobian IK Algorithm

while (($\mathbf{g}$ - $\mathbf{e}$) > Threshold):

  Compute $\mathbf{J}(\mathbf{e}, \mathbf{q})$ for the current configuration $\mathbf{q}$

  Compute $\mathbf{J}^{-1}$

  $\Delta\mathbf{e} = \alpha(\mathbf{g} - \mathbf{e})$ // choose a step to take

  $\Delta\mathbf{q} = \mathbf{J}^{-1}\Delta\mathbf{e}$ // compute required change in joints

  $\mathbf{q} = \mathbf{q} + \Delta\mathbf{q}$ // apply change to joints

  Compute resulting $\mathbf{e}$ // by FK

Denavit-Hartenberg (DH) Convention

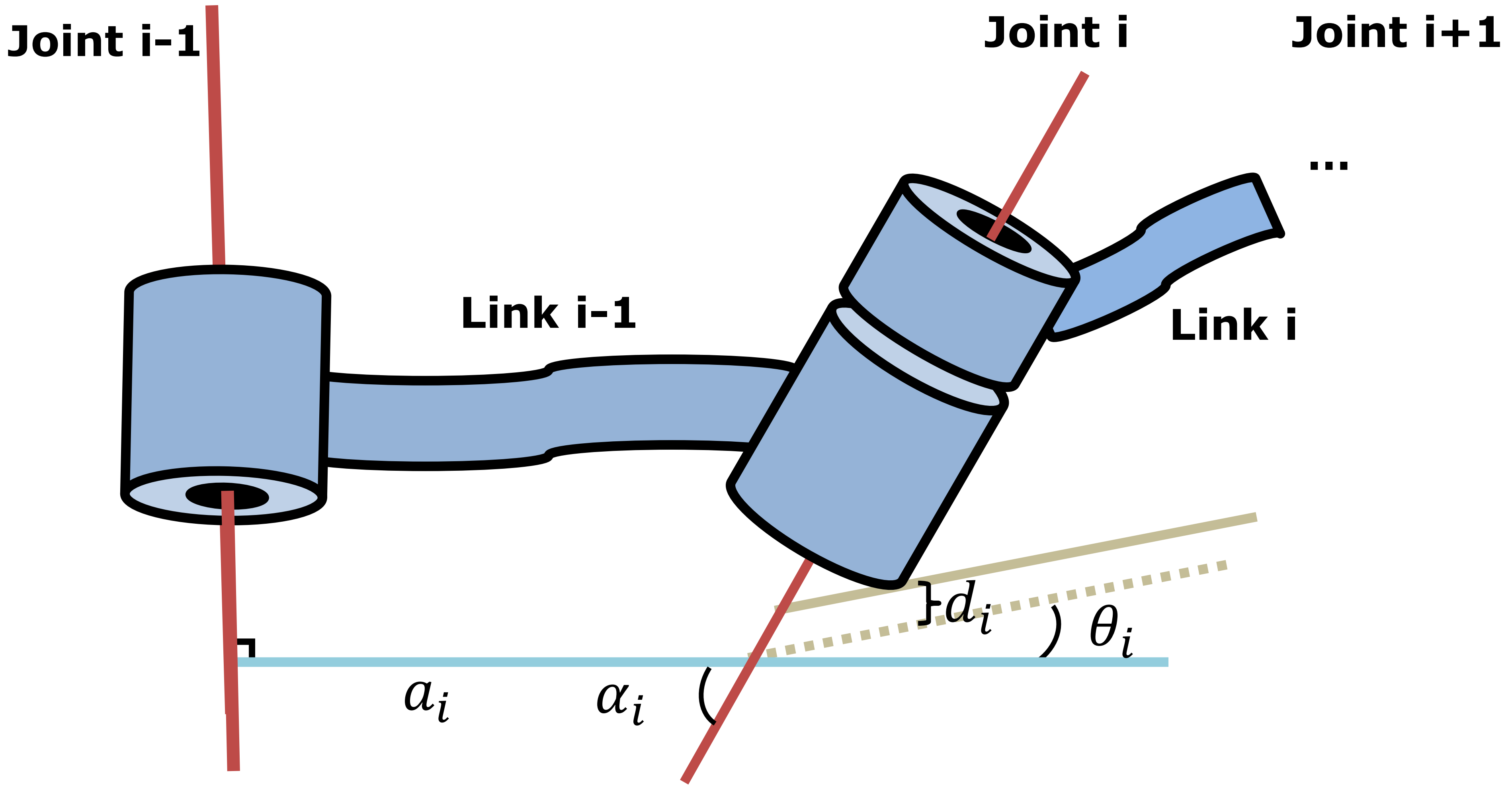
- **Goal:** Reduction of joint and link describing parameters
- **Approach:**
  - Systematic description of translation and rotation between neighbor links
  - Only 4 parameters

Denavit-Hartenberg Parameters

Describe joints via 4 DH-parameters:

- $\boldsymbol{a_i}$: **link length**, or distance between two consecutive *z*-axes, measured along the x_{i-1}-axis
- $\boldsymbol{\alpha_i}$: **link twist**, or angle between the *z*-axes of two consecutive joints measured about the x_{i-1}-axis
- $\boldsymbol{d_i}$: **link offset**, distance between two consecutive *x*-axes measured along z_i-axes
- $\boldsymbol{\theta_i}$: **joint angle** between two consecutive x-axes, measured about the z_i-axis

Denavit-Hartenberg Parameters



- α_i and a_i **describe the joint**
- d_i and θ_i **describe the connection to the next joint**

Denavit-Hartenberg Parameters

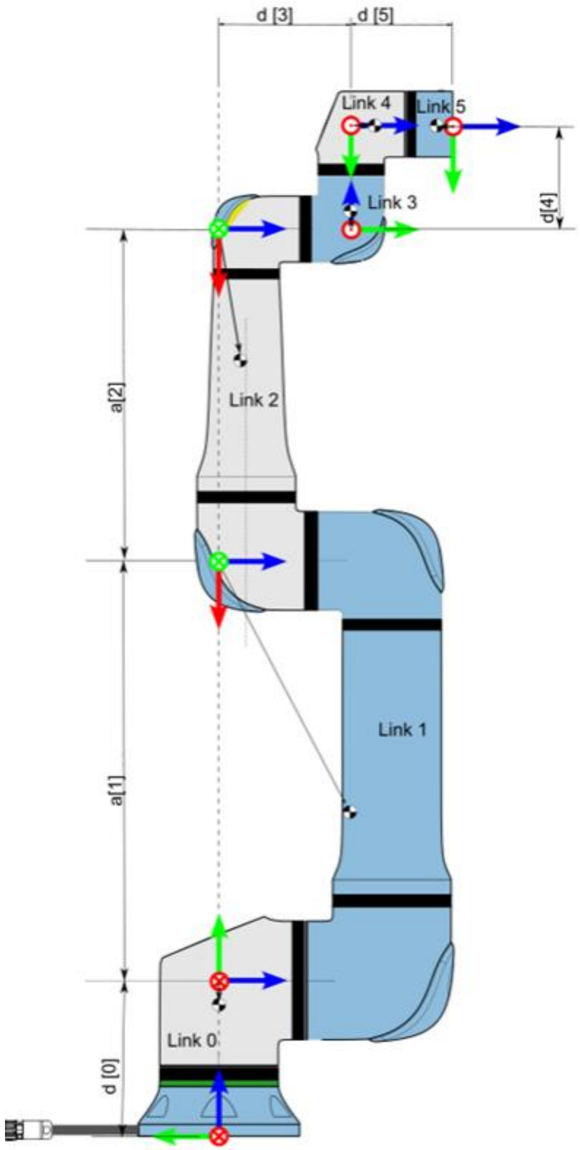
- Simplify computation of forward and inverse kinematics by using homogeneous transformations to describe each joint's effect to the next one

$$T_{n-1}^{n} = \begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i) * \cos(\alpha_i) & \sin(\theta_i) * sin(\alpha_i) & a_i * \cos(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i) * \cos(\alpha_i) & -\cos(\theta_i) * sin(\alpha_i) & a_i * sin(\theta_i) \\ 0 & sin(\alpha_i) & cos(\alpha_i) & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

J. Denavit and R. S. Hartenberg, A Kinematic Notation for Lower-Pair Mechanisms Based on Matrices, 1955

DH Example: UR5 Arm

UR5				
Kinematics	theta [rad]	a [m]	d [m]	alpha [rad]
Joint 1	0	0	0.089159	π/2
Joint 2	0	-0.425	0	0
Joint 3	0	-0.39225	0	0
Joint 4	0	0	0.10915	π/2
Joint 5	0	0	0.09465	-π/2
Joint 6	0	0	0.0823	0



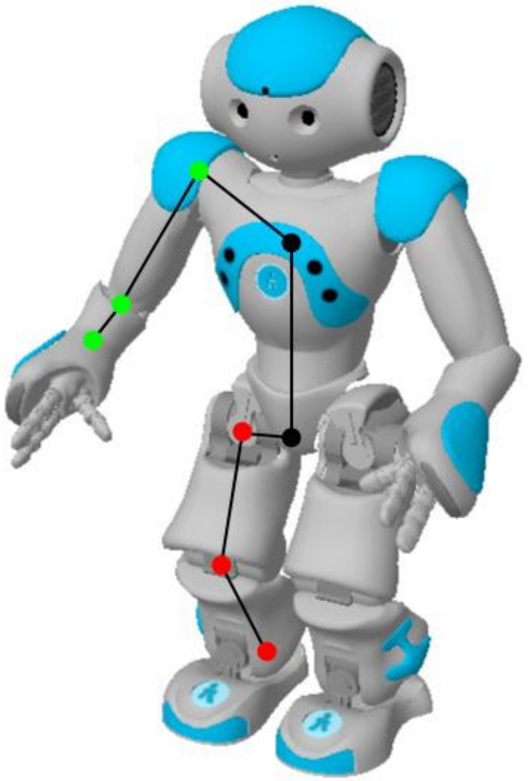
https://www.universal-robots.com/articles/ur/application-installation/dh-parameters-for-calculations-of-kinematics-and-dynamics/

Capability Maps

- Measures the **kinematic capabilities** of a robot, representing its workspace under certain quality measures
- Acceleration of online motion planning, due to **pre-calculation** of the capability measures
- **Common measures**:
  - Reachability
  - Manipulability
  - Robot base placeability (inverse reachability)
- Capability maps **always consist** of a reachability map

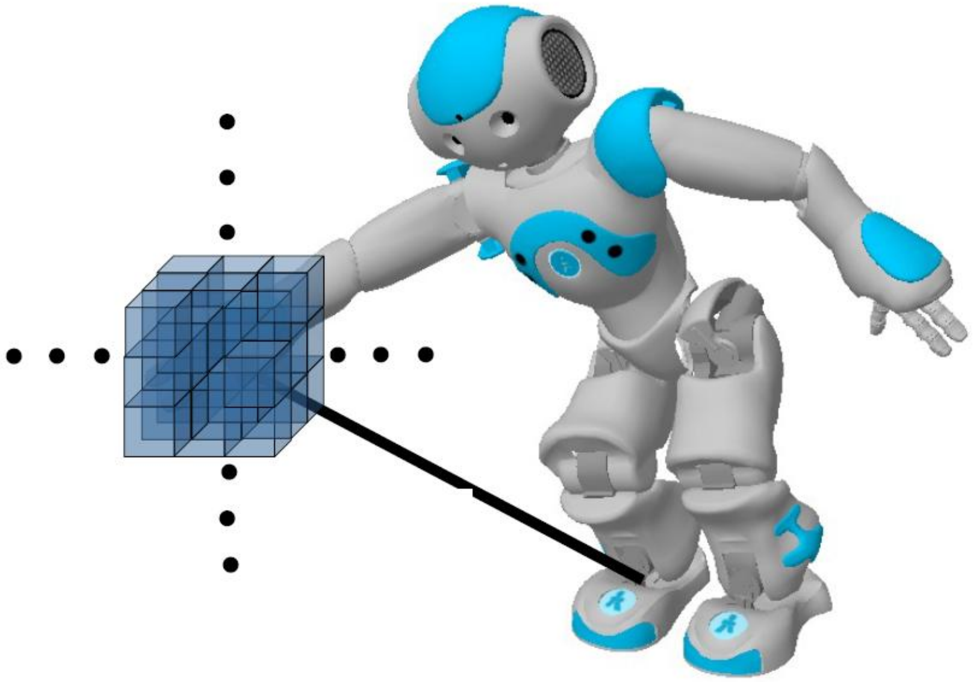
Reachability Map (RM)

- Constructed by systematic **sampling of joint configurations** of a kinematic chain
- **Example:** Chain of joints between the right foot and the gripper link



Reachability Map (RM)

- FK to determine the **corresponding voxel** containing the end-effector pose

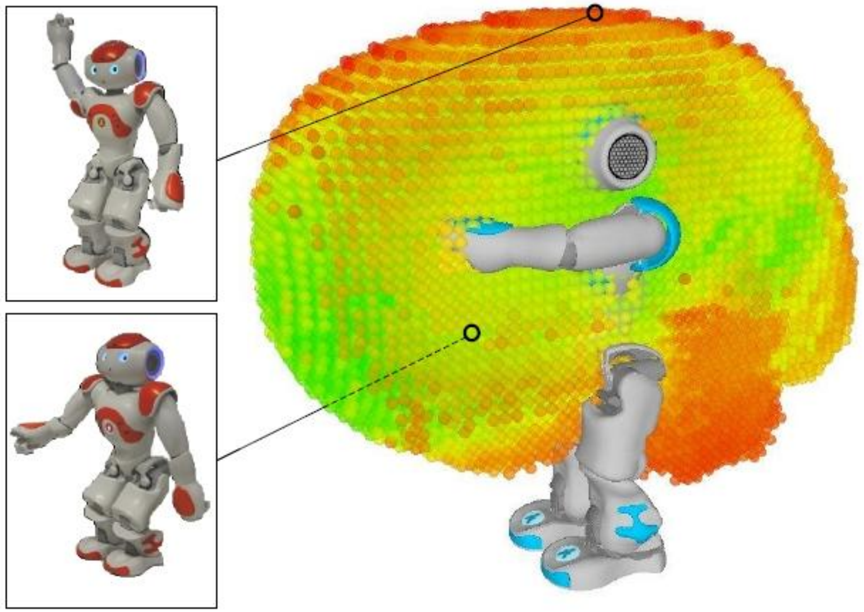


Reachability Map (RM)

- Configurations are added to the RM if they are **statically stable** and **self-collision free**
- Result: Representation of reachability, voxels contain configurations and **corresponding quality measures**
- Generating the RM is time-consuming, **but needs to be done only once offline**

Manipulability Measure

- Penalize configurations with limited maneuverability
- **Singular configurations:** Certain EE movements are not possible, i.e., **small desired changes** in EE poses lead to **large joint angle changes**
- **Consider:** Distance to singular configurations and joint limits, self-distance, ...



red=low
green=high

Inversion of the RM

- Invert the precomputed reachable workspace: **inverse reachability map (IRM)**
- Invert the FK transform for each configuration to get the **pose of the robot's base** (e.g., foot) **w.r.t. the EE pose**
- Store inverse transforms and manipulability measures
- Determine the **base pose voxel** in the IRM

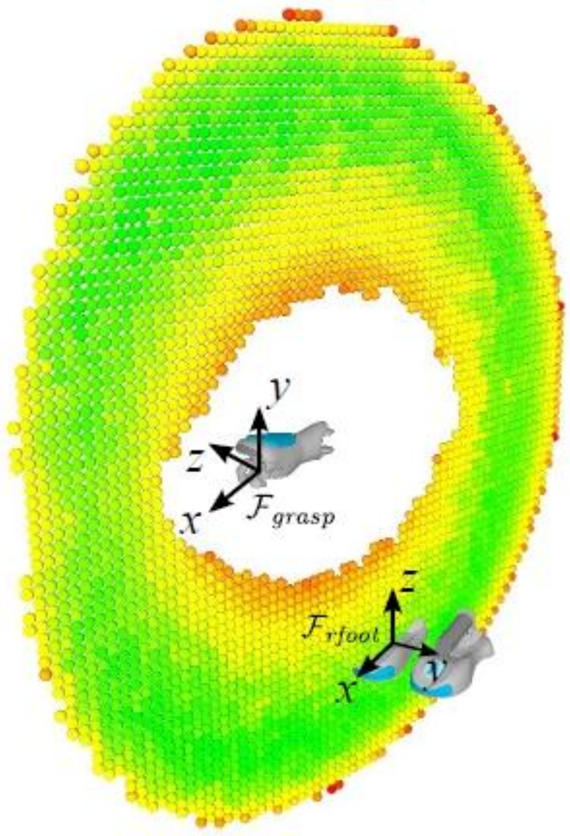
Inverse Reachability Map (IRM)

- The IRM represents the set of **potential base poses** relative to the EE frame
- Allows for selecting an optimal base pose for a given grasping target
- Can be pre-computed

red=low

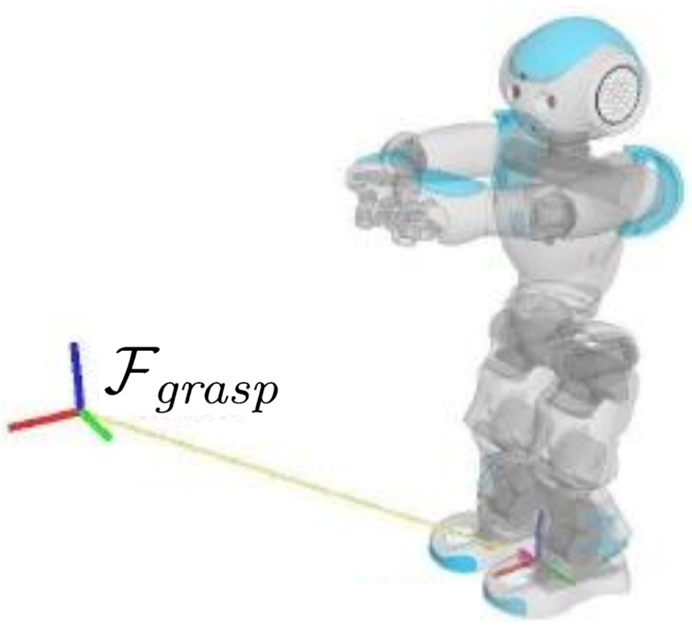
green=high

Cross section through the IRM showing potential feet locations



Determining the Optimal Stance Pose Given a Grasp Pose

- Given a desired 6D end-effector pose with transform $\mathcal{F}_{grasp}$
- How to determine the optimal stance pose?



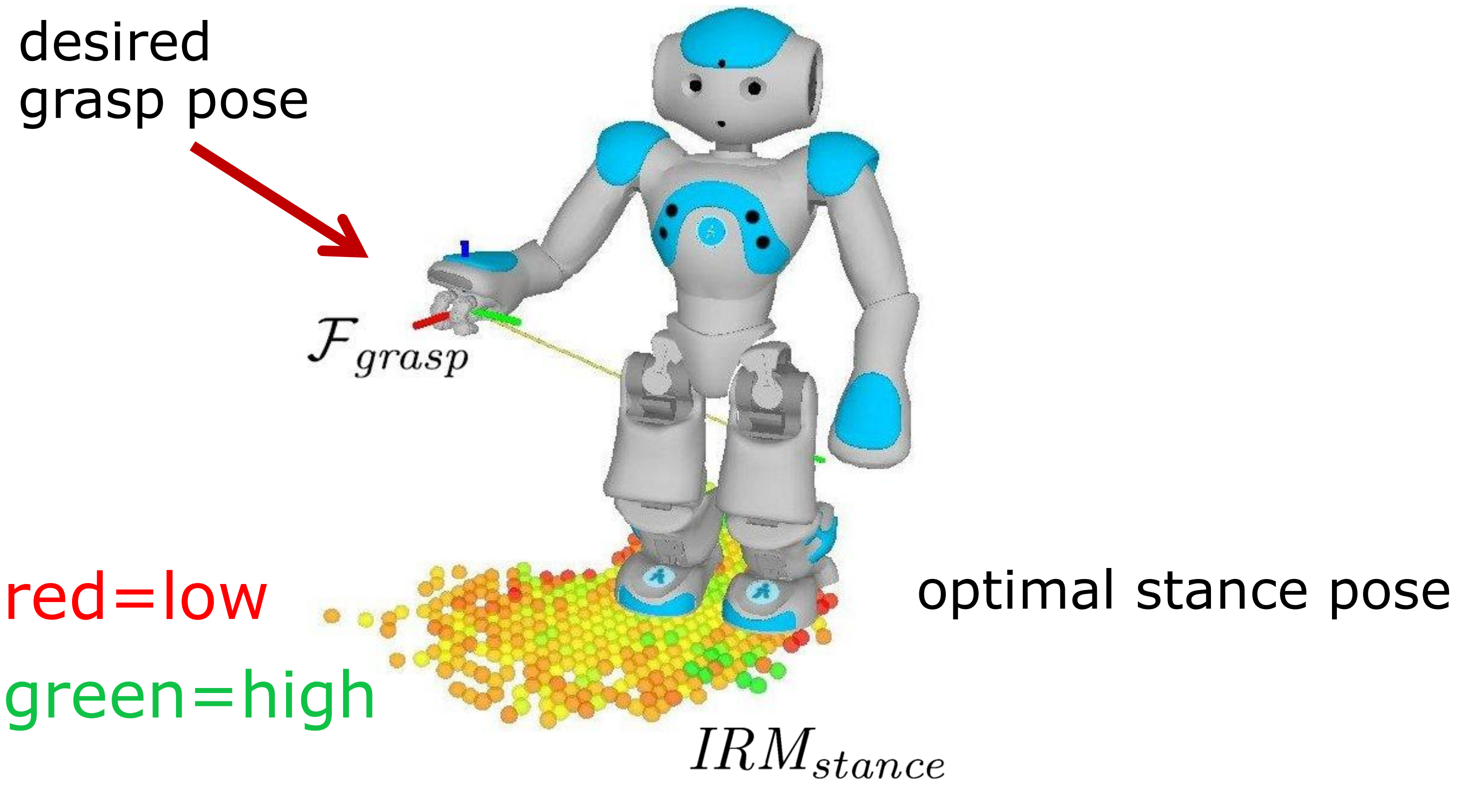
Determining the Optimal Stance Pose Given a Grasp Pose

- Transform the IRM and determine valid configurations of the feet on the ground
- Align the origin of the IRM with the grasp frame $\mathcal{F}_{grasp}$ to get the transformed IRM $tIRM$
- Intersect $tIRM$ with the floor plane *F*:

$$IRM_{floor} = tIRM \cap F$$

- Remove unfeasible configurations from IRM_{floor} to get IRM_{stance}

Determining the Optimal Stance Pose: Example



Select the optimal stance pose
from the voxel with the highest manipulability measure

Summary (1)

- Forward kinematics compute the end-effector pose given joint angles along the chain
- Inverse kinematics computes the joint angles so that the end-effector reaches a desired goal pose
- Several approaches for IK exist (analytical/numerical)
- Basic Jacobian IK technique iteratively adapts the joint angles to reach the end-effector goal pose
- Denavid-Hartenberg parameters reduce the joint and link describing parameters

Summary (2)

- Capability maps represent “how well” the robot can interact with the world
- Reachability maps represent reachable end-effector poses using FK
- Inverse RMs used to determine the optimal base pose for a desired end-effector pose
- Both computed offline only once

Literature

- Introduction to Inverse Kinematics with Jacobian Transpose, Pseudoinverse and Damped Least methods
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  John J. Craig, Pearson Prentice Hall, 2005
- Stance Selection for Humanoid Grasping Tasks by Inverse Reachability Maps
  F. Burget and M. Bennewitz,
  Proc. of the IEEE International Conference on Robotics & Automation (ICRA), 2015